

Formation control with binary information

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Abstract

In this paper, we study the problem of formation keeping of a network of strictly passive systems when very coarse information is exchanged. We assume that neighboring agents only know whether their relative position is larger or smaller than the prescribed one. This assumption results in very simple control laws that direct the agents closer or away from each other and take values in finite sets. We show that the task of formation keeping while tracking a desired trajectory and rejecting matched disturbances is still achievable under the very coarse information scenario. In contrast with other results of practical convergence with coarse or quantized information, here the control task is achieved exactly.

1 Introduction

Distributed motion coordination of mobile agents has attracted increasing attention in recent years owing to its wide range of applications from biology and social networks to sensor/robotic networks. Distributed formation keeping control is a specific case of motion coordination which aims at reaching a desired geometrical shape for the position of agents while tracking a desired velocity. This problem has been addressed with different approaches [32], [2], [29], [7], [24], [4], [30]. In problems of formation control, an important component, besides the dynamics of the agents and the graph topology, is the flow of information among the agents. In fact, the usual assumption in the literature on cooperative control is that a continuous flow of perfect information is exchanged among the agents. However, due to the coarseness of sensors and/or to communication constraints, the latter might be a restrictive requirement. To cope with the problem in the case of continuous-time agents' dynamics, the use of distributed quantized feedback control has been proposed in the literature [11], [18], [9], [10]. In fact, in formation control by quantized feedback control, information is transmitted among the agents whenever measurements cross the thresholds of the quantizer. At these times, the corresponding quantized value taken from a discrete set is transmitted. This allows to deal naturally both with the continuous-time nature of the agents' dynamics and with the discrete nature of the transmission information process without the need to rely on sampled-data models [9], [12], [17].

Literature review. The research on coordination in the presence of quantized or coarse information has mainly focused on discrete-time systems ([23], [19], [8], [26] to name a few). Motivated by the problem of reaching consensus in finite-time, [11] adopted binary control laws and cast the problem in the context of discontinuous control systems. The paper [10] proposed a consensus control algorithm in which the information collected from the neighbors is in binary form. The paper [18] has investigated a similar problem but in the presence of quantized measurements, while [9] has rigorously cast the problem in the framework of non-smooth control systems. The latter has also introduced a new class of hybrid quantizers to deal with possible chattering phenomena. Results on the formation control problem for continuous-time models of the agents in the presence of coarse information have also appeared. The authors of [34] study a rendezvous problem for Dubins cars based on a ternary feedback which depends on whether or not the neighbor's position is within the range of observation of the agent's sensor. Deployment on the line of a group of kinematic agents was shown to be achievable by distributed feedback which uses binary information control [14]. In [12], [17] formation control problems for groups of agents with second-order non-linear dynamics have been studied in the presence of quantized measurements. Leader-follower coordination for single-integrator agents and constant disturbance rejection by proportional-integral controllers with quantized information and time-varying topology has been studied in [33]. In this paper, we consider nonlinear agents and we adopt internal model based controllers to track reference velocities and reject constant and harmonic disturbances. On the other hand, we restrict ourselves to static graph topologies.

Main contribution. In this paper, we study the problem of distributed position-based formation keeping of a group of agents with strictly-passive dynamics which exchange binary information. The binary information models a

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sensing scenario in which each agent detects whether or not the components of its current distance vector from a neighbor are above or below the prescribed distance and apply a force (in which each component takes a binary value) to reduce or respectively increase the actual distance. A similar coarse sensing scenario was considered in [34] in the context of the so-called “minimalist” robotics.

Remarkably, despite such a coarse information and control action, we show that the control law guarantees exact achievement of the desired formation. This is an interesting result, since statically quantized control inputs typically generates practical convergence, namely the achievement of an approximate formation in which the distance from the actual desired formation depends on the quantizer resolution [12]. Here the use of binary information allows us to conclude asymptotic convergence without the need to dynamically update the quantizer resolution. The use of binary information in coordination problems ([11, 10]) has been proven useful to the design and real-time implementation of distributed controls for systems of first- or second-order agents in a cyber-physical environment (see e.g. [31, 27, 15]). We envision that a similar role will be played by the results in this paper for a larger class of coordination problems (see [16] for an early result in this respect).

Another advantage of our approach is that the resulting control laws are implemented by very simple directional commands (such as “move north”, “move north-east”, etc. or “stay still”). We also show that the presence of coarse information does not affect the ability of the proposed controllers to achieve the formation in a leader-follower setting in which the prescribed reference velocity is only known to the leader. This paper adopts a similar setting as in [12, 17] but controllers and analysis are different. Moreover, the paper investigates the formation control problem with unknown reference velocity tracking and matched disturbance rejection that was not considered in [12, 17]. Compared with [34], where also coarse information was used for rendezvous, the results in our contribution apply to a different class of systems and to a different cooperative control problem. Early results with the same sensing scenario but for a formation of agents modeled as double integrators have been presented in [22].

The paper is organized as follows: Section 2 introduces the problem statement along with some motivations and notation. Analysis of the formation keeping problem with coarse data in the case of known/unknown reference velocity is studied in Section 3. Section 4 investigates the problem of formation keeping with coarse data in the presence of matched disturbances. Related simulations are presented in Section 5. The paper is summarized in Section 6.

Notation. Given two sets S_1, S_2 , the symbol $S_1 \times S_2$ denotes the Cartesian product of two sets. This can be iterated. The symbol $\times_{k=1}^m S_k$ denotes $S_1 \times S_2 \times \dots \times S_m$. For a set S , $\text{card}(S)$ denotes the cardinality of the set S . Given a matrix M of real numbers, we denote by $\mathcal{R}(M)$ and $\mathcal{N}(M)$ the range and the null space, respectively. The symbols $\mathbf{1}, \mathbf{0}$ denotes vectors or matrices of all 1 and 0 respectively. I_p is the $p \times p$ identity matrix. Sometimes the size of the matrix is explicitly given. Thus, $\mathbf{1}_n$ is the n -dimensional vector of all 1. Given two matrices A, B , the symbol $A \otimes B$ denotes the Kronecker product.

2 Preliminaries

2.1 The multi-agent system

In this subsection we review the passivity-based approach to multi-agent system control ([2], see also [5, 25, 17]). A network of N agents in \mathbb{R}^p are considered. For each agent i , $x_i \in \mathbb{R}^p$ represents its position. The communication topology of the network is assumed to be modeled by a connected and undirected graph $G = (V, E)$, where V is a set of N nodes and $E \subseteq V \times V$ is a set of M edges connecting the nodes. Label one end of each edge in E with a positive sign and the other end with a negative sign. We define the relative position z_k as following

$$z_k = \begin{cases} x_i - x_j & \text{if node } i \text{ is the positive end of the edge } k \\ x_j - x_i & \text{if node } j \text{ is the positive end of the edge } k \end{cases}$$

where $x_i \in \mathbb{R}^p$ is the position of the agent i expressed in an inertial frame. We define the $N \times M$ incidence matrix B associated with the graph G as follows

$$b_{ik} = \begin{cases} +1 & \text{If node } i \text{ is the positive end of the edge } k \\ -1 & \text{If node } j \text{ is the positive end of the edge } k \\ 0 & \text{otherwise.} \end{cases}$$

By definition of B , we can represent the relative position variable z , with $z \triangleq [z_1^T \dots z_M^T]^T$, $z \in \mathbb{R}^{pM}$, as a function of the position variable x , namely

$$z = (B^T \otimes I_p)x. \quad (1)$$

Equation (1), implies that z belongs to the range space $\mathcal{R}(B^T \otimes I_p)$.

Each agent is expected to track its desired (time-varying) reference velocity denoted by v_i^r . Define the velocity error

$$y_i = \dot{x}_i - v_i^r. \quad (2)$$

The dynamics of each agent is given by

$$\mathcal{H}_i : \begin{cases} \dot{\xi}_i = f_i(\xi_i) + g_i(\xi_i)u_i \\ y_i = h_i(\xi_i) \end{cases} \quad (3)$$

where $\xi_i \in \mathbb{R}^{n_i}$ is the state variable, $u_i \in \mathbb{R}^p$ is the control input, $y_i \in \mathbb{R}^p$ is the velocity error, and the exogenous signal $v_i^r \in \mathbb{R}^p$ is the reference velocity for the agent i . The maps f_i , g_i and h_i are assumed to be locally Lipschitz such that $f_i(\mathbf{0}) = 0$, $g_i(\mathbf{0})$ is full column-rank, and $h_i(\mathbf{0}) = 0$.

The system \mathcal{H}_i is assumed to be strictly passive from the input u_i to the velocity error y_i . Since the system \mathcal{H}_i is strictly passive, there is a continuously differentiable storage function $S_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ which is positive definite and radially unbounded and satisfies

$$\frac{\partial S_i}{\partial \xi_i} [f_i(\xi_i) + g_i(\xi_i)u_i] \leq -W_i(\xi_i) + y_i^T u_i \quad (4)$$

where W_i is a continuous positive function and $W_i(\mathbf{0}) = 0$.

For the sake of conciseness, the equations (2) and (3) are written in the compact form

$$\begin{aligned} \dot{x} &= \underbrace{\begin{pmatrix} h_1(\xi_1) \\ \vdots \\ h_N(\xi_N) \end{pmatrix}}_{h(\xi)} + \underbrace{\begin{pmatrix} v_1^r \\ \vdots \\ v_N^r \end{pmatrix}}_{v^r} \\ \dot{\xi} &= \underbrace{\begin{pmatrix} f_1(\xi_1) \\ \vdots \\ f_N(\xi_N) \end{pmatrix}}_{f(\xi)} + \underbrace{\begin{pmatrix} g_1(\xi_1) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & g_N(\xi_N) \end{pmatrix}}_{g(\xi)} u \\ y &= h(\xi). \end{aligned} \quad (5)$$

We define the concatenated vectors $x \triangleq [x_1^T \dots x_N^T]^T$, $x \in \mathbb{R}^{pN}$, $x_i \in \mathbb{R}^p$, and $\xi \triangleq [\xi_1^T \dots \xi_N^T]^T$, $\xi \in \mathbb{R}^{pN}$, $\xi_i \in \mathbb{R}^p$. The desired position x^* , with $x^* \triangleq [x_1^{*T} \dots x_N^{*T}]^T$, $x^* \in \mathbb{R}^{pN}$, $x_i^* \in \mathbb{R}^p$ is given by $x^* = x_0^* + \mathbf{1}_N \otimes \int_0^t v^*(s)ds$, with x_0^* a constant vector. Analogously, $z^* \triangleq [z_1^{*T} \dots z_M^{*T}]^T$, $z^* \in \mathbb{R}^{pM}$, $z_i^* \in \mathbb{R}^p$, is the desired relative position vector. The two vectors are related by the identity $z^* = (B^T \otimes I_p)x^*$.

2.2 Problem Statement

Our goal is to design coordination control laws to attain the following collective behaviors for the formation of agents (2), (3):

- (i) The velocity of each agent of the network asymptotically converges to the desired reference velocity $v^*(t) \in \mathbb{R}^p$

$$\lim_{t \rightarrow \infty} \|\dot{x}_i(t) - v^*(t)\| = 0, \quad i = 1, \dots, N. \quad (6)$$

- (ii) Each relative position vector z_k converges to the desired relative position vector z_k^*

$$\lim_{t \rightarrow \infty} \|z_k - z_k^*\| = 0, \quad z_k^* \in \mathbb{R}^p, \quad k = 1, \dots, M. \quad (7)$$

- (iii) In the presence of matched input disturbances, i.e. in the case in which the equation (3) is replaced by

$$\dot{\xi}_i = f_i(\xi_i) + g_i(\xi_i)(u_i + d_i)$$

where d_i is a disturbance signal generated by a suitable exosystem (see Section 4), the collective behaviors in (i) and (ii) are still guaranteed.

Differently from other work in formation control, we are interested in control laws that achieve complex coordination tasks using very coarse information about the relative positions of the agents. We assume that the agents of the network are equipped with sensors that are capable to detect whether the relative position of two agents is above or below a prescribed one.

In line with our goals, for each agent i , the control law is designed as

$$u_i = - \sum_{k=1}^M b_{ik} \text{sign}(z_k - z_k^*), \quad i = 1 \dots N, \quad (8)$$

where $\text{sign}(\cdot)$ is the sign function and operates on each element of the p -dimensional vector $z_k - z_k^*$. The sign function $\text{sign}(\cdot) : \mathbb{R} \rightarrow \{-1, +1\}$ is defined as follows:

$$\text{sign}(\zeta) = \begin{cases} +1 & \text{if } \zeta \geq 0 \\ -1 & \text{if } \zeta < 0. \end{cases}$$

Observe that, $b_{ik} \neq 0$ if and only if edge k connects agent i to one of its neighbors. This implies the control law u_i uses only the information available to agent i . The above control law is inspired by the passivity-based control design proposed in [2].

Discussion about the control (8). The proposed control (8) has the following interpretation. Let the position of the agents be given with respect to an inertial frame in \mathbb{R}^p . Assign to each agent a local Cartesian coordinate system with an orientation identical to the inertial frame such that the current position of the agent is the origin of its local coordinate system. Therefore, there are p mutually perpendicular hyperplanes intersecting at the origin of the agent's local coordinate system and partitioning the state space into 2^p hyperoctants. For each neighbor of agent i , the corresponding p -dimensional vector $z_k - z_k^*$ lies in one of these hyperoctants (possibly at the boundary) and contributes the term $-b_{ik}\text{sign}(z_k - z_k^*)$ to the control u_i .

The implementation of (8) only requires the agent to detect when $z_k - z_k^*$ is crossing the boundary between hyperoctants.¹ A few other advantages of this sensing scenario are discussed below:

- (i) The formation is achieved with large inaccuracies in the measurements of $z_k - z_k^*$. As a matter of fact, the calculated control vector corresponding to a given vector $-b_{ik}(z_k - z_k^*)$ is the same no matter where $(z_k - z_k^*)$ precisely lies in the hyperoctant.
- (ii) If agent i detects that the vector $z_k - z_k^*$ is crossing the boundary of any two hyperoctants, then a new control vector is applied due to the term $-b_{ik}\text{sign}(z_k - z_k^*)$ in (8). In our analysis below, however, we show that the exact formation is still achieved if this control vector at the boundary takes any value in the convex hull of the two control vectors associated with the two hyperoctants. In this respect the control (8) is robust to possible uncertainties in the precise detection of the boundary crossing.
- (iii) The proposed control law guarantees the achievement of a desired formation by very simple directional commands, namely move closer if the actual position is larger than the desired one, move away if smaller.
- (iv) The input (8) provides a *finite*-valued control that uses *binary* measurements. In the case of networks of kinematic agents, the finite-valued control law of [11] inspired the design of self-triggered control algorithms (see [15]). These algorithms are shown to reduce the amount of time during which sensors are active and to reduce the amount of information exchanged among the agents. Moreover, they can be used to overcome possible fast switching of the controllers (see discussion in Section 5). The control investigated in this paper can pave the way towards the design of self-triggered cooperative control of nonlinear systems. A first step in this regard has been taken in [16].

3 Analysis

In this section we investigate the stability properties of the closed-loop system. We define the variable $\tilde{z} = z - z^*$ and write the control laws u_i , $i = 1, 2, \dots, N$, in the following compact form

$$u = -(B \otimes I_p) \text{sign } \tilde{z}. \quad (9)$$

From equation (1), we can write $\tilde{z} = (B^T \otimes I_p)(x - x^*)$. We consider the stability of the origin of the error system, with state variables $[\xi^T \ \tilde{z}^T]^T$. To derive the error system equations, we take the derivative of \tilde{z} , thus obtaining

$$\dot{\tilde{z}} = (B^T \otimes I_p)(\dot{x} - \dot{x}^*). \quad (10)$$

¹In that regard, the control law is reminiscent of the controllers of ([34]) that changes when a neighbor enters or leaves the field-of-view sector of the agent.

From equation (5), we further have

$$\dot{x} - \dot{x}^* = h(\xi) + v^r - \mathbf{1}_N \otimes v^* \quad (11)$$

where v^* is the desired reference velocity for the formation. By a property of the incidence matrix of a connected graph, $(B^T \otimes I_p)(\mathbf{1}_N \otimes v^*) = \mathbf{0}$. Thus, we represent the dynamics of the error system in the following general form

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)\dot{x} = (B^T \otimes I_p)(h(\xi) + v^r) \\ \dot{\xi} &= f(\xi) + g(\xi)(-B \otimes I_p)\text{sign } \tilde{z}. \end{aligned} \quad (12)$$

The system (12) has a discontinuous right-hand side due to the discontinuity of the sign function at zero. Before analyzing the system, we first define an appropriate notion of the solution. In this paper, the solutions to the system above are intended in the Krasowskii sense.

As in [9, 17], the motivation to consider these solutions lies in the fact that there exists convenient Lyapunov methods to analyze their asymptotic behavior and that they include other notions of solutions such as Carathéodory solutions if the latter exist. Let $X = (\tilde{z}, \xi)$ and let $F(X)$ be the set-valued map

$$F(X) = \left(\begin{array}{c} (B^T \otimes I_p)(h(\xi) + v^r) \\ f(\xi) \end{array} \right) - \left(\begin{array}{c} \mathbf{0}_{Np \times 1} \\ g(\xi)(B \otimes I_p) \end{array} \right) \mathcal{K} \text{sign } \tilde{z} \quad (13)$$

where $\mathcal{K} \text{sign } \tilde{z} = \times_{k=1}^M \times_{i=1}^p \mathcal{K} \text{sign } \tilde{z}_{ki}$ and

$$\mathcal{K} \text{sign } \tilde{z}_{ki} = \begin{cases} \{\text{sign } \tilde{z}_{ki}\} & \text{if } \tilde{z}_{ki} \neq 0 \\ [-1, 1] & \text{if } \tilde{z}_{ki} = 0. \end{cases} \quad (14)$$

We define $X(t) = (\tilde{z}(t), \xi(t))$ a Krasowskii solution to (12) on the interval $[0, t_1]$ if it is an absolutely continuous function which satisfies the differential inclusion

$$\dot{X}(t) \in F(X(t)) \quad (15)$$

for almost every $t \in [0, t_1]$, with F defined as in (13), and

$$\mathcal{K}(F(X)) = \bigcap_{\delta > 0} \overline{\text{co}}(F(B(x, \delta))), \quad (16)$$

with $\overline{\text{co}}$ the closed convex hull of a set and $B(x, \delta)$ the ball centered at x and of radius δ . Local existence of Krasowskii solutions to the differential inclusion above is always guaranteed (see e.g. [20] and [9], Lemma 3). For the convenience of the reader, a brief reminder about non-smooth control theory tools adopted in this paper is given in Appendix A.

3.1 Known Reference Velocity

In this section, we consider the case where the desired reference velocity, $v^*(t)$ is known to all the agents, namely $v^r = \mathbf{1}_N \otimes v^*$. Then, the closed-loop system simplifies as

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)h(\xi) \\ \dot{\xi} &= f(\xi) + g(\xi)(-B \otimes I_p)\text{sign } \tilde{z}. \end{aligned} \quad (17)$$

Proposition 1 Any Krasowskii solution to (17) exists for all $t \geq 0$ and converges to the origin.

Proof: The proof is based on the application of the non-smooth La Salle's invariance principle ([3]). Consider the following locally Lipschitz Lyapunov function

$$V(\tilde{z}, \xi) = \|\tilde{z}\|_1 + S(\xi)$$

where $\|\tilde{z}\|_1$ denotes the 1-norm $\|\tilde{z}\|_1 = \sum_{k,j} |\tilde{z}_{kj}|$ and evaluate its set-valued derivative $\dot{V}(\tilde{z}, \xi)$ along (17). We have

$$\dot{V}(\tilde{z}, \xi) = \{a \in \mathbb{R} : \exists w \in F(\tilde{z}, \xi) \text{ s.t. } a = \langle w, p \rangle, \text{ for all } p \in \partial V(\tilde{z}, \xi)\}.$$

By the definition of $F(\tilde{z}, \xi)$ in (13), for any $w \in F(\tilde{z}, \xi)$ there exists $w^{\tilde{z}} \in \mathcal{K}\text{sign } \tilde{z}$ such that

$$w = \begin{pmatrix} (B^T \otimes I_p)h(\xi) \\ f(\xi) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi) (B \otimes I_p) \end{pmatrix} w^{\tilde{z}}.$$

Observe that the Clarke generalized gradient $\partial V(\tilde{z}, \xi)$ is given by

$$\partial V(\tilde{z}, \xi) = \{p : p = \begin{pmatrix} p^{\tilde{z}} \\ \nabla S(\xi) \end{pmatrix} \text{ s.t. } p^{\tilde{z}}_{ki} \in \begin{cases} \{\text{sign } \tilde{z}_{ki}\} & \text{if } \tilde{z}_{ki} \neq 0 \\ [-1, +1] & \text{if } \tilde{z}_{ki} = 0 \end{cases} \}.$$

Suppose that $\dot{\bar{V}}(\tilde{z}, \xi) \neq \emptyset$ and take $a \in \dot{\bar{V}}(\tilde{z}, \xi)$. Then by definition there exists $w \in F(\tilde{z}, \xi)$ such that $a = \langle w, p \rangle$ for all $p \in \partial V(\tilde{z}, \xi)$. Choose $p \in \partial V(\tilde{z}, \xi)$ such that $p^{\tilde{z}} = w^{\tilde{z}}$. Thus

$$\begin{aligned} a &= \left\langle \begin{pmatrix} (B^T \otimes I_p)h(\xi) \\ f(\xi) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi) (B \otimes I_p) \end{pmatrix} w^{\tilde{z}}, \begin{pmatrix} w^{\tilde{z}} \\ \nabla S(\xi) \end{pmatrix} \right\rangle \\ &= \langle (B^T \otimes I_p)h(\xi), w^{\tilde{z}} \rangle + \langle f(\xi) - g(\xi)(B \otimes I_p)w^{\tilde{z}}, \nabla S(\xi) \rangle \\ &\leq -W(\xi) + \langle h(\xi), (B \otimes I_p)w^{\tilde{z}} \rangle + \langle (B^T \otimes I_p)h(\xi), w^{\tilde{z}} \rangle \\ &= -W(\xi) \end{aligned}$$

Hence, for any state (\tilde{z}, ξ) such that $\dot{\bar{V}}(\tilde{z}, \xi) \neq \emptyset$, $\dot{\bar{V}}(\tilde{z}, \xi) = \{a \in \mathbb{R} : a \leq -W(\xi)\} \subseteq (-\infty, 0]$. Therefore, the solutions can be extended for all $t \geq 0$ and by La Salle's invariance principle they converge to the largest weakly invariant set where $\xi = \mathbf{0}$. Since $v^r = \mathbf{1}_N \otimes v^*$, from (13), any point $(\tilde{z}, \mathbf{0})$ on this invariant set must necessarily satisfy

$$\mathbf{0} = \begin{pmatrix} \mathbf{0} \\ g(\mathbf{0})(B \otimes I_p) \end{pmatrix} w^{\tilde{z}}$$

for some $w^{\tilde{z}} \in \mathcal{K}\text{sign } \tilde{z}$. Since $g(\mathbf{0})$ is full-column rank, this implies $w^{\tilde{z}} \in \mathcal{N}(B \otimes I_p)$. Bearing in mind that $\tilde{z} \in \mathcal{R}(B^T \otimes I_p)$, then $\langle \tilde{z}, w^{\tilde{z}} \rangle = 0$. We claim that necessarily $\tilde{z} = \mathbf{0}$. In fact, by contradiction, $\tilde{z} \neq \mathbf{0}$ would imply the existence of at least a component $\tilde{z}_{ki} \neq 0$. As $\tilde{z}_{ki} \neq 0$ necessarily implies $w^{\tilde{z}}_{ki} \neq 0$ by definition of $w^{\tilde{z}} \in \mathcal{K}\text{sign } \tilde{z}$, then it would imply $\tilde{z}_{ki}w^{\tilde{z}}_{ki} > 0$, which shows the contradiction. Hence, on the invariant set, $\tilde{z} = \mathbf{0}$ and all the system's solutions converge to the origin. \square

3.2 Unknown reference velocity

In the previous section, we assumed that the desire reference velocity is known to all the agents. This assumption is very restrictive and one wonders whether it is possible to relax this assumption despite the very coarse information scenario we are confining ourselves. The positive answer to this question is given in the analysis below.

The problem of the reference velocity unknown to the agents is tackled here in the scenario in which at least one of the agents is aware of v^* . We refer to this agent as the leader ([29], [24]). As in [4], we further assume that the class of reference velocities that the formation can achieve are those generated by an autonomous system that we refer to as the exosystem. Given two matrices Φ, Γ^v , whose properties will be made precise later on, the exosystem obeys the following equations

$$\dot{w}^v = \Phi w^v, \quad v^* = \Gamma^v w^v. \quad (18)$$

Taking inspiration from the theory of output regulation (see e.g. [21]), an internal-model-based controller can be adopted for each agent $i = 2, 3, \dots, N$, namely

$$\begin{aligned} \dot{\eta}_i &= \Phi \eta_i + G \tilde{u}_i \\ v_i^r &= \Gamma^v \eta_i, \quad i = 2, 3, \dots, N. \end{aligned} \quad (19)$$

When $\tilde{u}_i = \mathbf{0}$ and the system is appropriately initialized, the latter system is able to generate any w^v solution to (18). Because v^* is known to agent 1, there is no need to implement system (19) at agent 1. The input \tilde{u}_i , as well as the input matrix G are to be designed later. Define $v^r = (v^{*T} \ v_2^{rT} \ \dots \ v_n^{rT})$ and the new error variables

$$\tilde{\eta}_i = \eta_i - w^v, \quad \tilde{v}_i = v_i^r - v^*, \quad i = 2, \dots, N,$$

along with the fictitious variables $\tilde{\eta}_1 = \mathbf{0}$, $\tilde{v}_1 = \mathbf{0}$. We obtain

$$\begin{aligned}\dot{\tilde{\eta}} &= \begin{pmatrix} \dot{\tilde{\eta}}_1 \\ \dot{\tilde{\eta}}_2 \\ \vdots \\ \dot{\tilde{\eta}}_n \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Phi & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Phi \end{pmatrix}}_{\tilde{\Phi}} \tilde{\eta} + \underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & G & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & G \end{pmatrix}}_{\tilde{G}} \underbrace{\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_n \end{pmatrix}}_{\tilde{u}} \\ \tilde{v} &= \underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Gamma^v & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Gamma^v \end{pmatrix}}_{\tilde{\Gamma}^v} \tilde{\eta}\end{aligned}$$

and in compact form

$$\begin{aligned}\dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta} + \tilde{G} \tilde{u} \\ \tilde{v} &= \tilde{\Gamma}^v \tilde{\eta}, \quad i = 1, 2, 3, \dots, N.\end{aligned}\tag{20}$$

From (12), the error position vector can be written as

$$\begin{aligned}\dot{\tilde{z}} &= (B^T \otimes I_p)(h(\xi) + v^r) \\ &= (B^T \otimes I_p)(h(\xi) + v^r - \mathbf{1}_n \otimes v^* + \mathbf{1}_n \otimes v^*) \\ &= (B^T \otimes I_p)(h(\xi) + \tilde{v}).\end{aligned}$$

The overall closed-loop system is

$$\begin{aligned}\dot{\tilde{z}} &= (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ \dot{\xi} &= f(\xi) + g(\xi)u \\ \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta} + \tilde{G} \tilde{u}.\end{aligned}\tag{21}$$

Proposition 2 Assume that Φ is skew-symmetric, namely

$$\Phi^T + \Phi = \mathbf{0},\tag{22}$$

(Γ^v, Φ) is an observable pair and let

$$G = \Gamma^{vT}.\tag{23}$$

Then all the Krasowskii solutions to (21) in closed-loop with the control input

$$u = \tilde{u} = -(B \otimes I_p) \text{sign } \tilde{z}\tag{24}$$

converge to the point $\tilde{z} = \mathbf{0}$, $\xi = \mathbf{0}$, $\tilde{\eta} = \mathbf{0}$.

Proof: Any Krasowskii solution to (21) with the control law defined by (24) satisfies a differential inclusion of the form (13) where $X = (\tilde{z}, \xi, \tilde{\eta})$ and

$$F(\tilde{z}, \xi, \tilde{\eta}) = \begin{pmatrix} (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ f(\xi) \\ \tilde{\Phi} \tilde{\eta} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \\ \tilde{G}(B \otimes I_p) \end{pmatrix} \mathcal{K} \text{sign } \tilde{z}.$$

Let

$$V(\tilde{z}, \tilde{v}, \tilde{\eta}) = \|\tilde{z}\|_1 + S(\xi) + \frac{1}{2} \tilde{\eta}^T \tilde{\eta}.$$

For any $w \in F(\tilde{z}, \xi, \tilde{\eta})$, there exists $w^{\tilde{z}} \in \mathcal{K} \text{sign } \tilde{z}$ such that

$$w = \begin{pmatrix} (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ f(\xi) \\ \tilde{\Phi} \tilde{\eta} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \\ \tilde{G}(B \otimes I_p) \end{pmatrix} w^{\tilde{z}}.$$

Moreover for any $p \in \partial V(\tilde{z}, \xi, \tilde{\eta})$ there exists $p^{\tilde{z}} \in \partial \|\tilde{z}\|_1$, with

$$p_{ki}^{\tilde{z}} \in \begin{cases} \{\text{sign } \tilde{z}_{ki}\} & \text{if } \tilde{z}_{ki} \neq 0 \\ [-1, +1] & \text{if } \tilde{z}_{ki} = 0, \end{cases}$$

and $k = 1, 2, \dots, m, i = 1, 2, \dots, p$, such that

$$p = \begin{pmatrix} p^{\tilde{z}} \\ \nabla S(\xi) \\ \tilde{\eta} \end{pmatrix}.$$

For $w^{\tilde{z}} \in \mathcal{K} \text{sign } \tilde{z}$, let $p \in \partial V(\tilde{z}, \xi, \tilde{\eta})$ such that $w^{\tilde{z}} = p^{\tilde{z}}$. Hence

$$\langle w, p \rangle = \langle (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}), w^{\tilde{z}} \rangle + \langle f(\xi) - g(\xi)(B \otimes I_p)w^{\tilde{z}}, \nabla S(\xi) \rangle + \langle \tilde{\Phi} \tilde{\eta} - \tilde{G}(B \otimes I_p)w^{\tilde{z}}, \tilde{\eta} \rangle.$$

By the conditions (22) and (23), the equality above simplifies as

$$\langle w, p \rangle = \langle (B^T \otimes I_p)h(\xi), w^{\tilde{z}} \rangle + \langle f(\xi) - g(\xi)(B \otimes I_p)w^{\tilde{z}}, \nabla S(\xi) \rangle.$$

Since the dynamic of each agent is strictly passive, we have

$$\langle f(\xi) - g(\xi)(B \otimes I_p)w^{\tilde{z}}, \nabla S(\xi) \rangle \leq -W(\xi) - \langle h(\xi), (B \otimes I_p)w^{\tilde{z}} \rangle$$

which implies $\langle w, p \rangle \leq -W(\xi)$.

Hence, similarly to the proof of Proposition 1, $\dot{\bar{V}}(\tilde{z}, \xi, \tilde{\eta}) = \{a \in \mathbb{R} : a \leq -W(\xi)\} \subseteq (-\infty, 0]$, provided that $\bar{V}(\tilde{z}, \xi, \tilde{\eta}) \neq \emptyset$. Because V is positive definite and proper, solutions are bounded and one can apply La Salle's invariance principle, which shows that every Krasowskii solution converges to the largest weakly invariant set where $W(\xi) = 0$. By definition of $W(\xi)$, the latter is equivalent to have convergence to the largest weakly invariant set where $\xi = 0$. To characterize this set, we look for points $(\tilde{z}, 0, \tilde{\eta})$ such that $(\dot{\tilde{z}}, 0, \dot{\tilde{\eta}}) \in F(\tilde{z}, 0, \tilde{\eta})$. Having $f(0) = h(0) = 0$, this means that we seek points $(\tilde{z}, 0, \tilde{\eta})$ for which there exists $w^{\tilde{z}} \in \mathcal{K} \text{sign } \tilde{z}$ such that

$$\begin{pmatrix} \dot{\tilde{z}} \\ 0 \\ \dot{\tilde{\eta}} \end{pmatrix} = \begin{pmatrix} (B^T \otimes I_p)(\tilde{\Gamma}^v \tilde{\eta}) \\ 0 \\ \tilde{\Phi} \tilde{\eta} \end{pmatrix} - \begin{pmatrix} 0 \\ g(0)(B \otimes I_p) \\ (\tilde{\Gamma}^v)^T (B \otimes I_p) \end{pmatrix} w^{\tilde{z}}.$$

The equality $0 = -g(0)(B \otimes I_p)w^{\tilde{z}}$, implies that necessarily $\tilde{z} = 0$ (see the final part of the proof of Proposition 1).

This also implies that on the invariant set the system evolves as

$$\begin{aligned} 0 &= (B^T \otimes I_p)(\tilde{\Gamma}^v \tilde{\eta}) \\ \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta}. \end{aligned}$$

From this point on, the argument is the same as in [4]. Indeed, as $\mathcal{N}(B^T) = \mathcal{R}(\mathbf{1}_n)$, by the block diagonal structure of $\tilde{\Gamma}^v$, we obtain that all the sub vectors of $\tilde{\Gamma}^v \tilde{\eta}$ must be the same and equal to zero, since the first p component of $\tilde{\Gamma}^v \tilde{\eta}$ are identically zero. In other words, $\Gamma^v \tilde{\eta}_i = 0$ for all $i = 2, 3, \dots, N$. Hence we conclude that on the largest weakly invariant set where $\xi = 0$, we have

$$\begin{aligned} 0 &= \Gamma^v \tilde{\eta}_i \\ \dot{\tilde{\eta}}_i &= \Phi \tilde{\eta}_i, \quad i = 2, 3, \dots, N. \end{aligned}$$

By the observability of (Γ^v, Φ) , it holds that $\tilde{\eta}_i = 0$ for all $i = 2, 3, \dots, N$. This finally proves the claim. \square

4 Formation control with matched disturbance rejection

In this section, we study the problem of formation keeping with very coarse exchanged information in the presence of matched input disturbances.

We consider a formation of strictly passive agents, where the dynamics of each agent of the networked system is

$$\dot{x}_i = y_i + v_i^r(t)$$

$$\mathcal{H}_i : \begin{cases} \dot{\xi}_i = f_i(\xi_i) + g_i(\xi_i)(u_i + d_i) \\ y_i = h_i(\xi_i), \end{cases} \quad (25)$$

where d_i is a disturbance. The network should converge to the prescribed formation and evolve with the given desired reference velocity v^* despite the action of the input disturbance d . As in the previous section, we first consider the case in which the reference velocity is unknown to all the agents except one. As explained in Section 3.2, we adopt an internal-model-based controller to recover the desired reference velocity. Similarly we suppose that the disturbance signal d_i at agent i is generated by an exosystem of the form

$$\dot{w}_i^d = \Phi_i^d w_i^d, \quad d_i = \Gamma_i^d w_i^d, \quad i = 1, 2, \dots, N. \quad (26)$$

To counteract the effect of the disturbance, we introduce an additional internal model controller given by

$$\begin{aligned} \dot{\theta}_i &= \Phi_i^d \theta_i + G_i^d \tilde{u}_i \\ \hat{d}_i &= \Gamma_i^d \theta_i, \end{aligned} \quad i = 1, 2, \dots, N, \quad (27)$$

where G_i^d, \tilde{u}_i will be designed later. We remark that the internal model for disturbance rejection is implemented also at agent 1 (the leader).

To both compensate the input disturbance and achieve the desired formation, the proposed control law is composed of two parts: \hat{u} guides the system to the desired formation, and \hat{d} compensates the input disturbance. Therefore, we can write the dynamics of each agent in the following compact form

$$\begin{aligned} u &= \hat{u} - \hat{d} \\ \dot{\xi} &= f(\xi) + g(\xi)(\hat{u} - \hat{d} + d). \end{aligned} \quad (28)$$

Define the new variable $\tilde{d}_i = \hat{d}_i - d_i$ and $\tilde{\theta}_i = \theta_i - w_i^d$, then

$$\begin{aligned} \dot{\tilde{\theta}}_i &= \Phi_i^d \tilde{\theta}_i + G_i^d \tilde{u}_i \\ \tilde{d}_i &= \Gamma_i^d \tilde{\theta}_i. \end{aligned} \quad (29)$$

In compact form we write

$$\begin{aligned} \dot{\tilde{\theta}} &= \begin{pmatrix} \dot{\tilde{\theta}}_1 \\ \dot{\tilde{\theta}}_2 \\ \dots \\ \dot{\tilde{\theta}}_n \end{pmatrix} = \underbrace{\begin{pmatrix} \Phi_1^d & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Phi_2^d & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Phi_n^d \end{pmatrix}}_{\Phi^d} \tilde{\theta} + \underbrace{\begin{pmatrix} G_1^d & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & G_2^d & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & G_n^d \end{pmatrix}}_{G^d} \underbrace{\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \dots \\ \tilde{u}_n \end{pmatrix}}_{\tilde{u}} \\ \tilde{d} &= \underbrace{\begin{pmatrix} \Gamma_1^d & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Gamma_2^d & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Gamma_n^d \end{pmatrix}}_{\Gamma^d} \tilde{\theta}. \end{aligned} \quad (30)$$

From the equation (21) together with the equations (28), (30) the overall closed-loop system is

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ \dot{\xi} &= f(\xi) + g(\xi)(\hat{u} - \Gamma^d \tilde{\theta}) \\ \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta} + \tilde{G} \tilde{u} \\ \dot{\tilde{\theta}} &= \Phi^d \tilde{\theta} + G^d \tilde{u} \end{aligned} \quad (31)$$

The following is proven:

Proposition 3 Assume that Φ and Φ_i^d , for $i = 1, 2, \dots, N$ are skew-symmetric matrices. If

$$\begin{aligned} G &= (\Gamma^v)^T \\ G_i^d &= (\Gamma_i^d)^T, \end{aligned} \quad (32)$$

then all the Krasowskii solutions to (31) in closed-loop with the control input

$$\hat{u} = \tilde{u} = -(B \otimes I_p) \text{sign} \tilde{z}, \quad \tilde{u} = h(\xi) \quad (33)$$

are bounded and converge to the largest weakly invariant set for

$$\begin{pmatrix} \dot{\tilde{z}} \\ \mathbf{0} \\ \dot{\tilde{\eta}} \\ \dot{\tilde{\theta}} \end{pmatrix} \in \begin{pmatrix} (B^T \otimes I_p) \tilde{\Gamma}^v \tilde{\eta} \\ -(B \otimes I_p) \mathcal{K} \text{sign} \tilde{z} - \Gamma^d \tilde{\theta} \\ \tilde{\Phi} \tilde{\eta} - (\tilde{\Gamma}^v)^T (B \otimes I_p) \mathcal{K} \text{sign} \tilde{z} \\ \Phi^d \tilde{\theta} \end{pmatrix}, \quad (34)$$

such that $\xi = \mathbf{0}$.

Proof: Consider the Lyapunov function

$$V(\tilde{z}, \tilde{v}, \tilde{\eta}, \tilde{\theta}) = \|\tilde{z}\|_1 + S(\xi) + \frac{1}{2} \tilde{\eta}^T \tilde{\eta} + \frac{1}{2} \tilde{\theta}^T \tilde{\theta}.$$

Any Krasowskii solution to (31) with the control law defined by (33) satisfies a differential inclusion of the form (15) where $X = (\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$ and

$$F(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}) = \begin{pmatrix} (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ f(\xi) - g(\xi) \Gamma^d \tilde{\theta} \\ \tilde{\Phi} \tilde{\eta} \\ \Phi^d \tilde{\theta} + G^d h(\xi) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \\ \tilde{G}(B \otimes I_p) \\ \mathbf{0} \end{pmatrix} \mathcal{K} \text{sign} \tilde{z}.$$

For any $w \in F(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$, there exists $w^{\tilde{z}} \in \mathcal{K} \text{sign} \tilde{z}$ such that

$$w = \begin{pmatrix} (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ f(\xi) - g(\xi) \Gamma^d \tilde{\theta} \\ \tilde{\Phi} \tilde{\eta} \\ \Phi^d \tilde{\theta} + G^d h(\xi) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \\ \tilde{G}(B \otimes I_p) \\ \mathbf{0} \end{pmatrix} w^{\tilde{z}}.$$

Moreover for any $p \in \partial V(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$ there exists $p^{\tilde{z}} \in \partial \|\tilde{z}\|_1$, with

$$p_{\tilde{z}i}^{\tilde{z}} \in \begin{cases} \{\text{sign } \tilde{z}_{ki}\} & \text{if } \tilde{z}_{ki} \neq 0 \\ [-1, +1] & \text{if } \tilde{z}_{ki} = 0, \end{cases}$$

and $k = 1, 2, \dots, m, i = 1, 2, \dots, p$, such that

$$p = \begin{pmatrix} p^{\tilde{z}} \\ \nabla S(\xi) \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix}.$$

For $w^{\tilde{z}} \in \mathcal{K} \text{sign} \tilde{z}$, let $p \in \partial V(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$ such that $w^{\tilde{z}} = p^{\tilde{z}}$. Hence,

$$\begin{aligned} \langle w, p \rangle &= \langle (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}), w^{\tilde{z}} \rangle \\ &\quad + \langle f(\xi) - g(\xi)((B \otimes I_p)w^{\tilde{z}} + \Gamma^d \tilde{\theta}), \nabla S(\xi) \rangle \\ &\quad + \langle \tilde{\Phi} \tilde{\eta} - \tilde{G}(B \otimes I_p)w^{\tilde{z}}, \tilde{\eta} \rangle + \langle \Phi^d \tilde{\theta} + G^d h(\xi), \tilde{\theta} \rangle. \end{aligned}$$

By the assumption (32), the above result simplifies as

$$\begin{aligned} \langle w, p \rangle &= \langle (B^T \otimes I_p)h(\xi), w^{\tilde{z}} \rangle \\ &\quad + \langle f(\xi) - g(\xi)((B \otimes I_p)w^{\tilde{z}} + \Gamma^d \tilde{\theta}), \nabla S(\xi) \rangle \\ &\quad + \langle G^d h(\xi), \tilde{\theta} \rangle. \end{aligned} \quad (35)$$

Since the dynamic of each agent is strictly passive, we have

$$\langle f(\xi) - g(\xi)((B \otimes I_p)w^{\tilde{z}} + \Gamma^d \tilde{\theta}), \nabla S(\xi) \rangle \leq -W(\xi) - \langle h(\xi), (B \otimes I_p)w^{\tilde{z}} + \Gamma^d \tilde{\theta} \rangle.$$

When it is replaced in (35), it gives $\langle w, p \rangle \leq -W(\xi)$, in view of $\Gamma^d = (G^d)^T$. As in the previous proofs, this shows $\tilde{V}(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}) = \{a \in \mathbb{R} : a \leq -W(\xi)\}$ and hence boundedness of the solutions and their convergence to the largest weakly invariant set for the closed-loop system such that $\xi = \mathbf{0}$. This proves the thesis. \square

Proposition 3 only proves boundedness of the solution and convergence of the velocity of each agent to its own reference velocity ($\xi = \mathbf{0}$). To prove the convergence of the network to the desired formation and evolution of the network with the desired velocity, one should additionally prove that \tilde{z} converges to the origin. This result does not hold in general but there are a few cases of interest where this is true. In what follows we study two of these cases.

Case I: $\Phi = \mathbf{0}$, $\Phi_i^d = \mathbf{0}_{p \times p}$, $i = 1, 2, \dots, N$, the graph \mathcal{G} has no loops and $\tilde{\Gamma}^v$, $\tilde{\Gamma}_i^d$, $i = 1, 2, \dots, N$, are nonsingular.

This case correspond to the scenario in which the unknown reference velocity and the disturbances are constant signals. The arguments below show that the distributed control laws characterized in Proposition 3 guarantee the achievement of the desired formation and the prescribed velocity while rejecting the disturbances in the special case in which the edge Laplacian $B^T B$ of the graph is non-singular. The latter is true if the graph has no loops ([24]). From Proposition 3 it is known that the system converges to the largest weakly invariant set for (34) such that $\xi = \mathbf{0}$. In the case $\Phi = \mathbf{0}$, $\Phi_i^d = \mathbf{0}_{p \times p}$, this implies that for any state $(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}) = (\tilde{z}, \mathbf{0}, \tilde{\eta}, \tilde{\theta})$ on this set, and for any $w \in F(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$, there exists $w^{\tilde{z}} \in \mathcal{K} \text{ sign } \tilde{z}$, such that

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)(\tilde{\Gamma}^v \tilde{\eta}) \\ \mathbf{0} &= -(B \otimes I_p)w^{\tilde{z}} - \Gamma^d \tilde{\theta} \\ \dot{\tilde{\eta}} &= -(\tilde{\Gamma}^v)^T (B \otimes I_p)w^{\tilde{z}} \\ \dot{\tilde{\theta}} &= \mathbf{0}. \end{aligned} \tag{36}$$

From $\dot{\tilde{\theta}} = \mathbf{0}$, we conclude that $\tilde{\theta}$ is a constant vector. Hence, considering the second equality in (36), we conclude that $(B \otimes I_p)w^{\tilde{z}}$ is a constant vector. Bearing in mind that the graph is a tree, then the edge Laplacian $B^T B$ is invertible, and we infer that $w^{\tilde{z}}$ is also constant. From the third equation we deduce that each component of $\tilde{\eta}(t)$ evolves with law $\alpha t + \beta$. However, this would contradict the boundedness of the solutions. Hence, $\tilde{\eta}(t)$ must be constant. Hence, necessarily $(B \otimes I_p)w^{\tilde{z}} = \mathbf{0}$. Recalling the proof of the Proposition 1, the latter implies $\tilde{z} = \mathbf{0}$, and in particular $\dot{\tilde{z}} = \mathbf{0}$. Now, consider the first equality in (36) and multiply both sides by $B \otimes I_p$. Since $\dot{\tilde{z}} = \mathbf{0}$, we obtain $\mathbf{0} = (B B^T \otimes I_p)(\tilde{\Gamma}^v \tilde{\eta})$. From the property of the Laplacian of a connected undirected graph, we conclude $\tilde{\Gamma}^v \tilde{\eta} = \mathcal{R}(\mathbf{1}_N \otimes I_p)$. From the definition of $\tilde{\Gamma}^v \tilde{\eta}$, we have $\tilde{\Gamma}^v \tilde{\eta} = \mathbf{0}$ and hence $\tilde{\eta} = \mathbf{0}$. Moreover, $\Gamma^d \tilde{\theta} = \mathbf{0}$ shows that $\tilde{\theta} = \mathbf{0}$.

The arguments above allow us to infer the following:

Corollary 1 Assume that $\Phi = \mathbf{0}$ and $\Phi_i^d = \mathbf{0}$, for $i = 1, 2, \dots, N$. Also assume that Γ^v , $\tilde{\Gamma}_i^d$, $i = 2, 3, \dots, N$, are non-singular and the graph G has no loops. If

$$\begin{aligned} \tilde{G} &= (\tilde{\Gamma}^v)^T \\ G^d &= (\Gamma^d)^T, \end{aligned} \tag{37}$$

then all the Krasowskii solutions to the system (31) in closed-loop with the control input

$$\hat{u} = \tilde{u} = -(B \otimes I_p) \text{sign } \tilde{z}, \quad \check{u} = h(\xi) \tag{38}$$

are bounded and converge to the origin.

Remark 1 (Limitations due to coarse information) The assumption on the invertibility of Γ^v , $\tilde{\Gamma}_i^d$ is a mild one. In fact it is tantamount to assume that each velocity component must track a constant reference velocity and each input channel is affected by the disturbance.

The assumption on the absence of loops in the graph is a technical assumption needed to overcome the limitations due to the use of coarse measurements in the proof. In fact if $\text{sign } \tilde{z}$ is replaced by \tilde{z} , then in the proof one can use a different argument that does not require the invertibility of the edge Laplacian.

As a matter of fact, the second identity in (36) becomes $\mathbf{0} = -(B \otimes I_p)\tilde{z} - \Gamma^d \tilde{\theta}$, from which it follows that

$\mathbf{0} = -(B \otimes I_p)\dot{\tilde{z}}$. The latter and the identity $\dot{\tilde{z}} = (B^T \otimes I_p)\tilde{\Gamma}^v \tilde{\eta}$ yield that $\mathbf{0} = (BB^T \otimes I_p)\tilde{\Gamma}^v \tilde{\eta}$ and therefore $\mathbf{0} = \tilde{\Gamma}^v \tilde{\eta}$. From here the proof is similar to the previous one. In other words, the use of \tilde{z} allows immediately to infer that $\mathbf{0} = \tilde{\Gamma}^v \tilde{\eta}$ from which the proof can be continued without any further obstruction.

Case II: Harmonic disturbance rejection with known reference velocity.

We consider now the case in which the reference velocity is known and the controller only adopts an internal model to reject the disturbances. In this case, the system (31) becomes

$$\begin{aligned}\dot{\tilde{z}} &= (B^T \otimes I_p)h(\xi) \\ \dot{\tilde{\xi}} &= f(\xi) + g(\xi)(\hat{u} - \Gamma^d \tilde{\theta}) \\ \dot{\tilde{\theta}} &= \Phi^d \tilde{\theta} + G^d \tilde{u}.\end{aligned}\tag{39}$$

The choice $G^d = (\Gamma^d)^T$, $\hat{u} = -(B \otimes I_p)\text{sign} \tilde{z}$, $\tilde{u} = h(\xi)$ as designed in Proposition 3 yields that all the Krasowskii solutions to the closed-loop system

$$\begin{aligned}\dot{\tilde{z}} &= (B^T \otimes I_p)h(\xi) \\ \dot{\tilde{\xi}} &= f(\xi) + g(\xi)(-(B \otimes I_p)\text{sign} \tilde{z} - \Gamma^d \tilde{\theta}) \\ \dot{\tilde{\theta}} &= \Phi^d \tilde{\theta} + (\Gamma^d)^T h(\xi)\end{aligned}\tag{40}$$

are bounded and converge to the largest weakly invariant set for (40) such that $\xi = \mathbf{0}$. In the case of harmonic disturbances, disturbance rejection can be proven.

Corollary 2 If for $i = 1, 2, \dots, N$ the exosystems have matrices (Γ_i^d, Φ_i^d) of the form

$$\Phi_i^d = \text{block.diag} \left\{ \begin{pmatrix} 0 & \omega_{i1} \\ -\omega_{i1} & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \omega_{ip} \\ -\omega_{ip} & 0 \end{pmatrix} \right\},$$

with $\omega_{ij} \neq 0$ for all $j = 1, \dots, p$, and

$$\Gamma_i^d = \text{block.diag} \{ \Gamma_{i1}^d, \dots, \Gamma_{ip}^d \}$$

with $\Gamma_{ij}^d \neq \mathbf{0}$, for all $j = 1, \dots, p$, then all the Krasowskii solutions to (40) converge to the set of points where $\xi = \mathbf{0}$ and $\tilde{z} = \mathbf{0}$.

Proof: For the sake of notational simplicity, we present the proof in the case $p = 1$, but this assumption does not change the result for the general case $p > 1$.

A solution on the weakly invariant set for (40) is such that $(\tilde{z}, \tilde{\theta})$ satisfies

$$\begin{aligned}\dot{\tilde{z}} &= \mathbf{0} \\ \dot{\tilde{\theta}} &= \Phi^d \tilde{\theta}\end{aligned}\tag{41}$$

and there exists $w^{\tilde{z}} \in \mathcal{K} \text{sign} \tilde{z}$ such that $\mathbf{0} = g(\mathbf{0})(-(B \otimes I_p)w^{\tilde{z}} - \Gamma^d \tilde{\theta})$. Since by assumption $g(\mathbf{0})$ is full-column rank, the latter is equivalent to

$$\mathbf{0} = -(B \otimes I_p)w^{\tilde{z}} - \Gamma^d \tilde{\theta}.\tag{42}$$

From (42),

$$\sum_{k=1}^M b_{ik} w_k^{\tilde{z}} = -\Gamma_i^d \tilde{\theta}_i\tag{43}$$

where the term on the right-hand side is a sinusoidal function oscillating around zero. Considering the left-hand side of the equality (43), we define the following three types of links:

- (i) A-type: A link is of A-type if $\tilde{z}_k \neq 0$ and there is one other link $\tilde{z}_{k'} \neq 0$ connected to the same node such that $b_{ik} w_k^{\tilde{z}} + b_{ik'} w_{k'}^{\tilde{z}} = 0$. Note that the number of A-type links of each of the nodes of the graph is even.
- (ii) B-type: A link is of B-type if $\tilde{z}_k \neq 0$ and there is no other link connected to the same node that cancels the term $-b_{ik} w_k^{\tilde{z}}$.
- (iii) C-type: A link is of C-type if $\tilde{z}_k = 0$. Therefore, according to the Krasowski notion of solution, the term $b_{ik} w_k^{\tilde{z}}$ can take any value in the interval $[-1, 1]$.

Since $\dot{\tilde{z}} = \mathbf{0}$, \tilde{z} is a constant vector. We claim that the constraint (43) implies that the links of each node can be of C-type only. As a first step, we argue that a B-type link cannot exist. Assume, there are B-type links connected

to the node i . This would imply a constant term in the left-hand side of (43) that would contradict the equality (43) itself, since the right-hand side of the equality is a sinusoidal function oscillating around zero.

As a second step, we show that A-type links cannot exist. Assume by contradiction that there are A-type links connected to the node i . Since each link connects two nodes together, consider one of the A-type links connected to the node i and then consider the node at the other end of the link, that we label as $i + 1$. Now, consider the equality (43) for the node $i + 1$ and follow the following procedure:

1. If the A-type link of the node i is also an A-type link for the node $i + 1$, then go to step 3, else go to step 2.
2. Since a B-type link cannot exist, then the link is a C-type link and consequently $\tilde{z}_k = 0$. Go back to the node i . Since the link associated with \tilde{z}_k is a C-type link, the link associated with $\tilde{z}_{k'}$, that satisfied $b_{ik}w_k^{\tilde{z}} + b_{ik'}w_{k'}^{\tilde{z}} = 0$ is also a C-type link. Therefore, both links associated with $\tilde{z}_k, \tilde{z}_{k'}$ are C-type links. Now, go to step 1 and consider other A-type links of the node i .
3. Continue from the node $i + 1$. Consider any of its A-type links, then go to step 1 and repeat the procedure from the node $i + 1$.

At the end of this procedure, one proves that either there are no A-type links or that there is a loop \mathcal{L} in the graph of A-type links. Let \mathcal{B} be the incidence matrix associated with the loop. Then by construction

$$\mathcal{B}w^{\tilde{z}_{\mathcal{L}}} = \mathbf{0}$$

where $\tilde{z}_{\mathcal{L}}$ is the sub-vector of \tilde{z} associated with the loop \mathcal{L} . Since $\tilde{z}_{\mathcal{L}} \in \mathcal{R}(\mathcal{B}^T)$ and $w^{\tilde{z}_{\mathcal{L}}} \in \mathcal{N}(\mathcal{B})$, one can prove that $\tilde{z}_{\mathcal{L}} = \mathbf{0}$ as in the proof of Proposition 1. This contradicts the fact that \mathcal{L} is a loop of A-type links. We conclude that type-A links cannot exist. Hence, all the links must be of C-type and therefore $\tilde{z} = \mathbf{0}$. The argument indicates that on the invariant set where $\xi = \mathbf{0}$, \tilde{z} must be equal to $\mathbf{0}$. \square

Remark 2 (Limitations due to coarse information (Cont'd)) If we remove the limitation on the available measurements, i.e. if we replace $\text{sign}\tilde{z}$ with \tilde{z} , then the result above can be extended to exosystems having also eigenvalues at zero, thus allowing for disturbances that are a linear combination of constants and sinusoidal signals. In fact, if this is the case, the largest invariant set for (39) such that $\xi = \mathbf{0}$, includes the set of points for which the solution to (41) issuing from these points satisfies (cf. (42))

$$\mathbf{0} = -(B \otimes I_p)\tilde{z} - \Gamma^d\tilde{\theta}. \quad (44)$$

Moreover, \tilde{z} is a constant. Repeated differentiation of $\mathbf{0} = -(B \otimes I_p)\tilde{z} - \Gamma^d\tilde{\theta}$ implies

$$\begin{pmatrix} -\Gamma^d\Phi^d \\ -\Gamma^d(\Phi^d)^2 \\ \vdots \\ -\Gamma^d(\Phi^d)^\nu \end{pmatrix} \tilde{\theta} = \mathbf{0}. \quad (45)$$

In view of the diagonal structure of (Γ^d, Φ^d) (see (30)), it holds true that²

$$\begin{pmatrix} -\Gamma_i^d\Phi_i^d \\ -\Gamma_i^d(\Phi_i^d)^2 \\ \vdots \\ -\Gamma_i^d(\Phi_i^d)^\nu \end{pmatrix} \tilde{\theta}_i = \mathbf{0}.$$

Assuming observability of the pairs (Γ_i^d, Φ_i^d) , and that $\Phi_i^d \neq \mathbf{0}$, the matrix is full-column rank and $\tilde{\theta}_i = \mathbf{0}$. Since this is true for each i , then $\tilde{\theta} = \mathbf{0}$. From the identity $\mathbf{0} = -(B \otimes I_p)\tilde{z} - \Gamma^d\tilde{\theta}$, it descends that $\mathbf{0} = -(B \otimes I_p)\tilde{z}$. Since $\tilde{z} \in \mathcal{R}(B^T \otimes I_p)$, then necessarily $\langle \tilde{z}, \tilde{z} \rangle = 0$ from which it follows that $\tilde{z} = \mathbf{0}$. This shows that if \tilde{z} is fully available, then the controls in this section guarantees that the formation is achieved along with disturbance rejection and without any assumption on the exosystem matrices with the exception of the skew-symmetry and of the observability properties. These internal-model-based controllers appear to be more general than others discussed in the recent literature (see e.g. [1]). Other applications of internal-model-based controllers for disturbance rejection in networked systems can be found in [13, 6].

²The authors would like to thank Bayu Jayawardhana for a useful suggestion on this argument.

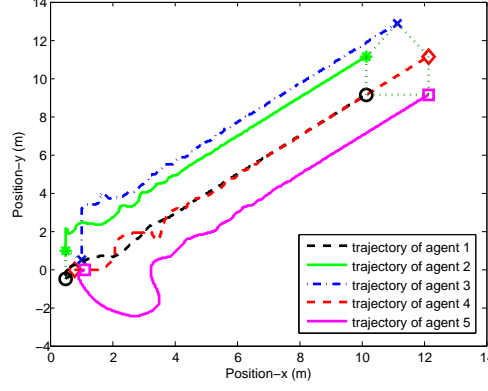


Figure 1: The evolution of five agents in \mathbb{R}^2 tracking a constant reference velocity only known to the leader. The system reaches the desired formation and exhibits a translational motion with the desired constant velocity.

5 Simulations

In this section we present the simulation results for a group of five strictly passive systems in \mathbb{R}^2 . The dynamics of each agent is given by

$$\mathcal{H}_i : \begin{cases} \dot{\xi}_i = -\xi_i + u_i \\ y_i = \xi_i \end{cases} \quad (46)$$

where, comparing with (3), $f_i(\xi_i) = -\xi_i$, $g_i(\xi_i) = I_2$, and $h_i(\xi_i) = \xi_i$. The agents exchange information over a connected graph. The associated incidence matrix is

$$B = \begin{pmatrix} +1 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The desired formation has a pentagon shape with edge length equal to 2 and is defined by the following inter-agent distance vectors: $z_1^* = [0 \ -2]^T$, $z_2^* = [-1 \ -1 - \sqrt{3}]^T$, $z_3^* = [-1 \ 1 + \sqrt{3}]^T$, $z_4^* = [0 \ 2]^T$. Note that the number of edges of the graph is four. The initial position of the agent is set to $x(0) = [0.5 \ -0.5 \ 0.5 \ 1 \ 1 \ 0.5 \ 0.8 \ 0 \ 1.1 \ 0]^T$. The simulation refers to the case of a formation evolving with a constant reference velocity known only to the formation leader (agent 1). The following are set as internal model parameters for all agents: $\Phi = \mathbf{0}_{2 \times 2}$, $\Gamma^v = I_2$, $G = I_2$, $w^v(0) = [0.6 \ 0.6]^T$, and $\eta_i(0) = [0 \ 0]^T$, for $i = 1, \dots, 5$. Figure 1 shows the evolution of the described system with the desired constant velocity $v^* = [0.6 \ 0.6]^T$. The other agents generate the reference velocity using the control laws based on the internal model principle. Figure 2 shows the time behavior of the horizontal component of \tilde{z}_1 , $\text{sign } \tilde{z}_1$ and the corresponding control \tilde{u}_1 . As time elapses, \tilde{z}_1 converges to the origin implying convergence to the desired relative position. While \tilde{z}_1 converges to the origin, $\text{sign } \tilde{z}_1$ and \tilde{u}_1 converge to the discontinuity surface and oscillate between $+1$ and -1 . The state variables associated with the other agents exhibit a similar behavior and are not shown. Figure 3 shows the horizontal and vertical components of the reference velocity v^* and the estimated velocities v_i^r . The leader generates a constant desired velocity and the follower agents estimate the same reference velocity after some time.

Simulation results in the presence of matched input disturbances are presented next. Figures 4 and 5 show the state of the system with harmonic input disturbance and known reference velocity (Case II). The desired reference velocity is $v^* = [0.6 \ 0.6]^T$ and is known to all of the agents. In this example, the following are set as the internal model parameters:

$\Phi_1^d = I_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Gamma_i^d = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0.5 \end{pmatrix}$, $G_i^d = \Gamma_i^{dT}$, $w_i^d(0) = [1 \ 1 \ 1 \ 1]^T$, $\Phi_2^d = 2\Phi_1^d$, $\Phi_1^d = \Phi_3^d = \Phi_5^d$, $\Phi_4^d = 0.5\Phi_1^d$. Figure 4 shows the results when the disturbance is tackled by a controller based on the internal model principle. The result confirms that both the desired formation and the desired velocity are asymptotically achieved. Figure 5 shows the time behavior of the horizontal component of \tilde{z}_1 , $\text{sign } \tilde{z}_1$ and the corresponding control \tilde{u}_1 . Similar to the case without the disturbance, while time elapses, \tilde{z}_1 converges to the

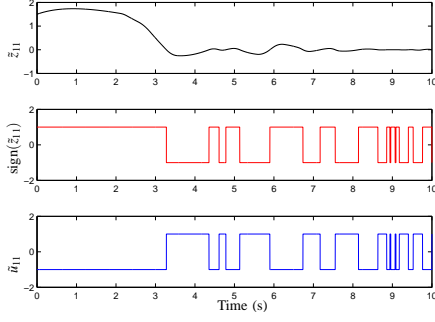


Figure 2: The plot of the (horizontal) x -component of \tilde{z}_1 , sign \tilde{z}_1 and the corresponding control \tilde{u}_1 .

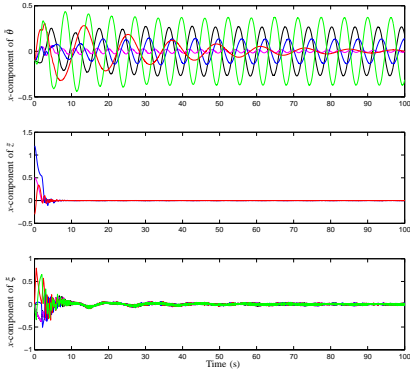


Figure 4: The horizontal component of the disturbance estimation error $\tilde{\theta}$ and the velocity error ξ for all five agents, together with the horizontal component of the relative position vector \tilde{z} . Here, an internal-model-based controller is used to reject the harmonic disturbance. ξ and \tilde{z} asymptotically converge to zero.

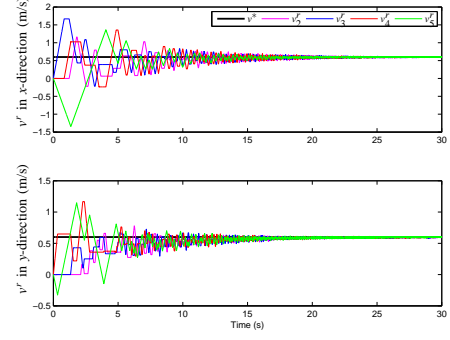


Figure 3: The horizontal and vertical component of the estimated reference velocity v_i^r for each agent i . The leader (agent 1) generates the desired velocity $v^* = [0.6 \ 0.6]^T$ (m/s).

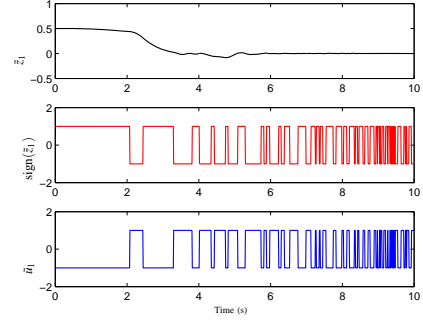


Figure 5: The plot of the (horizontal) x -component of \tilde{z}_1 , sign \tilde{z}_1 and the corresponding control \tilde{u}_1 in the presence of harmonic input disturbance.

origin implying convergence to the desired relative position. While \tilde{z}_1 converges to the origin, sign \tilde{z}_1 and \tilde{u}_1 converge to the discontinuity surface and oscillate between $+1$ and -1 . Figures 6 and 7 show the state $(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$ of the system with constant input disturbance and constant unknown reference velocity (Case I). The following are the parameters chosen for the simulation: $\Phi_i^d = \Phi = \mathbf{0}_{2 \times 2}$, $\Gamma^v = \Gamma_i^d = I_2$, $G = G_i^d = I_2$, $w^v(0) = [0.6 \ 0.6]^T$, $w_i^d(0) = [i \ i + 3]^T$ and $\eta_i(0) = [0 \ 0]^T$. The communication graph is a tree. The associated incidence matrix is

$$B_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ +1 & -1 & -1 & -1 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}.$$

Figure 6 shows the horizontal component of the input disturbance w^d and the velocity ξ (for all 5 agents), together with the reference velocity estimation error $\tilde{\eta}$ (for all follower agents) and the horizontal component of the relative position vector \tilde{z} . In this simulation no internal-model-based controller is used. As shown, the desired formation does not track the desired reference velocity. Figure 7 shows the results after implementing the internal-model-based controller. As predicted, the formation is achieved, the desired velocity is reached by all the agents and the input disturbances are rejected. Figure 8 shows the time behavior of the horizontal component of \tilde{z}_1 , sign \tilde{z}_1 and the corresponding control \tilde{u}_1 . As time evolves, \tilde{z}_1 converges to the origin, and sign \tilde{z}_1 and \tilde{u}_1 start switching

between $+1$ and -1 with high frequency. This behavior may not be acceptable in practice and can be overcome by the hysteric quantizers studied in [9] or the self-triggered controllers of [15].

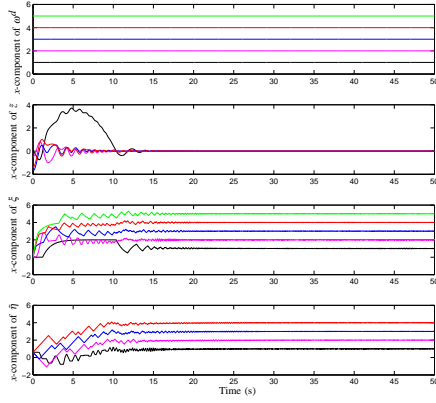


Figure 6: The horizontal component of the input disturbance w^d and the velocity ξ (for all 5 agents), together with the reference velocity estimation error $\tilde{\eta}$ (for follower agents) and the horizontal component of the relative position vector \tilde{z} . In this simulation, the disturbance is constant and no internal-model-based controller is used. As a result, the variables do not converge to zero.

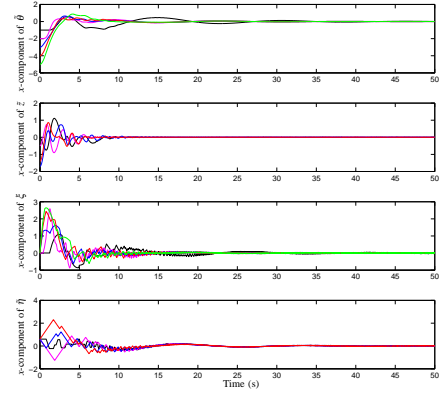


Figure 7: The horizontal component of the disturbance estimation error $\tilde{\theta}$ and the velocity ξ (for all 5 agents), together with the reference velocity estimation error $\tilde{\eta}$ (for follower agents) and the horizontal component of the relative position vector \tilde{z} . Here, the constant disturbance is rejected by the internal-model-based controller and ξ , $\tilde{\eta}$, $\tilde{\theta}$, \tilde{z} asymptotically converge to zero.

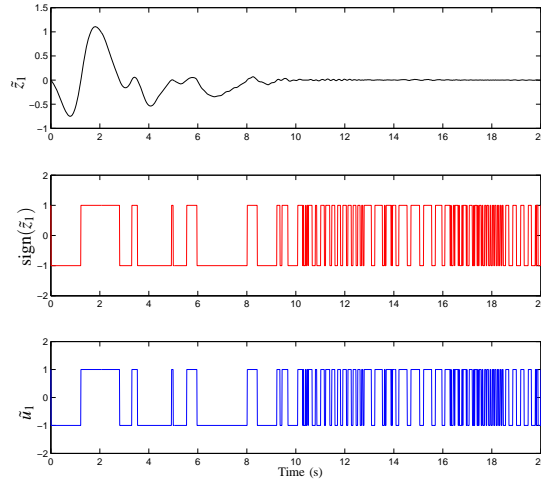


Figure 8: The plot of the (horizontal) x-component of \tilde{z}_1 , $\text{sign } \tilde{z}_1$ and the corresponding control \tilde{u}_1 in the presence of constant reference velocity and constant disturbance.

6 Conclusion

In this paper we considered a formation control problem with very coarse information for a network of strictly passive systems. We showed that despite the very coarse information, the exact formation is reached. Moreover, the formation tracks a desired reference velocity even in the case when the reference velocity is only available to one of the agents (the so-called leader). In the same coarse sensing scenario, we presented results of matched input disturbance rejection. We remarked that the use of coarse information poses a few limits to the formation control problem that can be solved.

Possible future avenues of research include the extension of the results to deal with time-varying topologies. A few related results have been discussed in [12, 33]. Moreover, discontinuous control laws as those considered in this paper can be viewed as the outcome of a non-smooth optimization problem associated with the original control problem ([5]) and it would be interesting to investigate this topic more in depth. Another interesting topic to understand better is whether the finite-valued control laws of this paper can be used to tackle the case of (asymmetric) measurement noise. Finally, we observe that as the system converges to the prescribed formation, fast oscillations of the control inputs between $+1$ and -1 may occur. As discussed in the simulation section, these oscillations could be overcome by the hysteresis quantizer of [9] or the self-triggered approach of [15]. A comprehensive treatment of this aspect is another interesting topic that deserves attention.

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A Non Smooth Control Theory Tools

This appendix is added for the convenience of the Reviewers only. It will be omitted from the final version of the manuscript.

We recall some basic notations from the theory of nonsmooth control which will be used throughout the paper. $x_0 \in \mathbb{R}^N$ is a Krasowskii equilibrium for the differential inclusion $\dot{x} \in F(x(t))$ if the function $x(t) = x_0$ is a Krasowskii solution to $\dot{x} \in F(x(t))$ starting from the initial condition x_0 , namely if $\mathbf{0} \in \mathcal{K}(F(x))$. A set \mathcal{S} is weakly (respectively, strongly) invariant for $\dot{x} \in F(x(t))$ if for any initial condition $\bar{x} \in \mathcal{S}$ at least one (all the) Krasowskii solution $x(t)$ starting from \bar{x} belongs (belong) to \mathcal{S} for all t in the domain of the definition of $x(t)$. Let V be a locally Lipschitz continuous function. Recall that by Rademacher's theorem, the gradient ∇V of V exists almost everywhere. Let R be the set of measure zero where $\nabla V(x)$ does not exist. Then the Clarke generalized gradient of V at x is the set $\partial V(x) = co\{\lim_{i \rightarrow +\infty} \nabla V(x_i) : x_i \rightarrow x, x_i \notin S, x_i \notin R\}$ where S is any set of measure zero in \mathbb{R}^N . We define the set-valued derivative of V at x with respect to (15) the set $\dot{V}(x) = \{a \in \mathbb{R} : \exists v \in \mathcal{K}(F(x)) \text{ s.t. } a = p.v, \forall p \in \partial V(x)\}$.

Non-smooth LaSalle's invariance principle. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz and regular function. Let $\bar{x} \in \mathcal{S}$, with \mathcal{S} compact and strongly invariant for $\dot{x} \in F(x(t))$. Assume that for all $\bar{x} \in \mathcal{S}$ either $\dot{V} = \emptyset$ or $\dot{V} \subseteq (-\infty, 0]$. Then any Krasowskii solution to $\dot{x} \in F(x)$ starting from \bar{x} converges to the largest weakly invariant subset contained in $\mathcal{S} \cap \{x \in \mathbb{R}^n : \mathbf{0} \in \dot{V}\}$, with $\mathbf{0}$ the null vector in \mathbb{R}^n .